

A NON-COMMUTATIVE AMIR-CAMBERN THEOREM FOR VON NEUMANN ALGEBRAS

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ABSTRACT. We prove that if two von Neumann algebras are sufficiently close in the Banach-Mazur cb-distance (up to a universal constant), then they are isomorphic. We also prove that the length of a C^* -algebra is stable under perturbations by cb-isomorphisms with small bound. Both results rely on the fact that almost completely isometric maps are almost multiplicative.

1. INTRODUCTION

This note concerns perturbation of operator algebras relative to the Banach-Mazur cb-distance. Recall that the Banach-Mazur cb-distance (or cb-distance in short) between two operator spaces \mathcal{X}, \mathcal{Y} is defined as:

$$d_{cb}(\mathcal{X}, \mathcal{Y}) = \inf\{\|T\|_{cb}\|T^{-1}\|_{cb}\},$$

where the infimum runs over all possible cb-isomorphisms $T : \mathcal{X} \rightarrow \mathcal{Y}$. This extends the classical Banach-Mazur distance for Banach spaces when these are endowed with their minimal operator space structure. For background on completely bounded maps and operator space theory the reader is referred to [2], [5], [10] and [13].

Recall that the Amir-Cambern Theorem (see [1], [3]) states that if the Banach-Mazur distance between two $C(K)$ -spaces is strictly smaller than 2, then these spaces are $*$ -isomorphic as C^* -algebras. As $C(K)$ -spaces are exactly unital commutative C^* -algebras, one is tempted to extend the Amir-Cambern Theorem to all unital C^* -algebras. Here, we prove a non-commutative Amir-Cambern Theorem for von Neumann algebras:

Theorem There exists $\varepsilon_0 > 0$ such that for any von Neumann algebras \mathcal{M}, \mathcal{N} , the inequality $d_{cb}(\mathcal{M}, \mathcal{N}) < 1 + \varepsilon_0$ implies that \mathcal{M} and \mathcal{N} are $*$ -isomorphic.

Note that our bound is universal but not explicit. The cb-distance concerns only the operator space structure, hence the basic idea is to

gain the algebraic structure. It is known that unital completely isometric bijections between operator algebras are necessarily multiplicative (see Theorem 4.5.13 [2]). Therefore, one wants to prove that almost completely isometric maps are almost multiplicative; we manage to prove this by an ultraproduct argument (see Theorem 2.1). When maps are between C^* -algebras, we can drop the ‘unital’ hypothesis and show that the unitization of an almost completely isometric map between C^* -algebras is almost multiplicative (see Corollary 2.5). Consequently, starting from a linear cb-isomorphism with small bound between von Neumann algebras, one can define a new multiplication (on one of the algebras) close to the original multiplication. Then, we use the fact that the vanishing of the second and third completely bounded Hochschild cohomology groups of an operator algebra over itself implies a strong stability property under perturbation by close multiplications (see Proposition 3.2). It is crucial to work with the completely bounded cohomology here, because we can use the important result that every completely bounded cohomology group of a von Neumann algebra over itself vanishes (see [15]), this is unknown for the bounded cohomology.

As another application of Corollary 2.5, we can prove that the length of a C^* -algebra is stable under cb-isomorphisms with small bound. The notion of length of an operator algebra has been defined by G. Pisier in [11] in order to attack the Kadison similarity problem. The length is an invariant for algebraic cb-isomorphisms. To prove Theorem 3.7, one has to notice that the length is stable for close multiplications. Actually, we have the following general principle: any property which is stable under perturbation by close multiplications is stable under perturbation by cb-isomorphisms with small bound.

2. ALMOST COMPLETELY ISOMETRIC MAPS

For $d \in \mathbb{N} \setminus \{0\}$, we let $T^{\vee d}$ denote the d -linear map defined on \mathcal{A}^d by:

$$T^{\vee d}(a_1, \dots, a_d) = T(a_1 \cdots a_d) - T(a_1) \cdots T(a_d).$$

For ultraproducts of operator spaces and completely bounded maps, see [13]. The ultraproduct argument used in the following proof is inspired from [7].

Theorem 2.1. *For each $d \in \mathbb{N} \setminus \{0\}$, for any $\eta > 0$, there exists $\rho \in (0, 1)$ such that for any unital operator algebras \mathcal{A}, \mathcal{B} , for any unital cb-isomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$, $\|T\|_{cb} \leq 1 + \rho$ and $\|T^{-1}\|_{cb} \leq 1 + \rho$ imply $\|T^{\vee d}\|_{cb} < \eta$.*

Proof. Suppose the assertion is false. Then there exists $\eta_o > 0$ such that for every positive integer $n \in \mathbb{N} \setminus \{0\}$, there is a unital cb-isomorphism $T_n : \mathcal{A}_n \rightarrow \mathcal{B}_n$ between some unital operator algebras satisfying $\|T_n\|_{cb} \leq 1 + 1/n$, $\|T_n^{-1}\|_{cb} \leq 1 + 1/n$ and $\|T_n^{\vee d}\|_{cb} \geq \eta_o$. Let \mathcal{U} be a non-trivial ultrafilter on \mathbb{N} , let us denote $\mathcal{A}_{\mathcal{U}}$ (resp. $\mathcal{B}_{\mathcal{U}}$) the ultraproduct $\Pi_n \mathbb{K}^1 \otimes_{\min} \mathcal{A}_n / \mathcal{U}$ (resp. $\Pi_n \mathbb{K}^1 \otimes_{\min} \mathcal{B}_n / \mathcal{U}$), here \mathbb{K}^1 denotes the unitization of the C^* -algebra of all compact operators on ℓ_2 . Then $\mathcal{A}_{\mathcal{U}}$ (resp. $\mathcal{B}_{\mathcal{U}}$) is a unital operator algebra (see [2]). Now consider $T_{\mathcal{U}} : \mathcal{A}_{\mathcal{U}} \rightarrow \mathcal{B}_{\mathcal{U}}$ the ultraproduct map obtained from the $\text{id}_{\mathbb{K}^1} \otimes T_n$'s. Hence $T_{\mathcal{U}}$ is a unital linear complete isometry between operator algebras, so $T_{\mathcal{U}}$ is multiplicative (see [2]). This contradicts the hypothesis for all n , $\|T_n^{\vee d}\|_{cb} \geq \eta_o$. Because $\|T_n^{\vee d}\|_{cb} = \|(\text{id}_{\mathbb{K}} \otimes T_n)^{\vee d}\|$, the condition $\|T_n^{\vee d}\|_{cb} \geq \eta_o$ means that there are $x_{1,n}, \dots, x_{d,n}$ in the closed unit ball of $\mathbb{K} \otimes_{\min} \mathcal{A}_n$ such that

$$\|(\text{id}_{\mathbb{K}} \otimes T_n)(x_{1,n} \cdots x_{d,n}) - (\text{id}_{\mathbb{K}} \otimes T_n)(x_{1,n}) \cdots (\text{id}_{\mathbb{K}} \otimes T_n)(x_{d,n})\| \geq \eta_o,$$

which implies that

$$\|T_{\mathcal{U}}(\dot{x}_1 \cdots \dot{x}_d) - T_{\mathcal{U}}(\dot{x}_1) \cdots T_{\mathcal{U}}(\dot{x}_d)\| \geq \eta_o$$

(where \dot{x}_i denotes the equivalence class of $(x_{i,n})_n$ in $\mathcal{A}_{\mathcal{U}}$). This last inequality contradicts the multiplicativity of $T_{\mathcal{U}}$. ■

Our purpose now is to state a non-unital version of the previous Theorem. We need to unitize cb-isomorphisms. The next Lemma is very interesting because it gives an operator space characterization of invertible elements in a C^* -algebra (The author thanks Éric Ricard for suggesting this result).

Lemma 2.2. *Let \mathcal{A} be a unital C^* -algebra and $x \in \mathcal{A}$, $\|x\| \leq 1$. Then, x is invertible if and only if there exists $\alpha > 0$ such that for any $y \in \mathcal{A}^{**}$ of norm one,*

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 \geq \alpha + \|y\|^2 \text{ and } \left\| \begin{bmatrix} x & y \end{bmatrix} \right\|^2 \geq \alpha + \|y\|^2 \quad (C)$$

*In this case, the maximum of the α 's satisfying (C) equals $\|x^{-1}\|^{-2}$ and moreover, condition (C) is actually satisfied for any $y \in \mathcal{A}^{**}$.*

Proof. Suppose x is non-invertible in \mathcal{A} . Then x is non-invertible in the von Neumann algebra \mathcal{A}^{**} . Let us suppose first that x is not left invertible in \mathcal{A}^{**} . Hence $\mathcal{A}^{**}x$, the w^* -closed left ideal generated by x in

\mathcal{A}^{**} , is proper. Consequently, there is a non-trivial projection $p \in \mathcal{A}^{**}$ such that $\mathcal{A}^{**}x = \mathcal{A}^{**}p$. Now define $y = 1 - p$, so

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \max\{\|x\|, \|y\|\} = 1 < \sqrt{\alpha + \|y\|^2}, \quad \forall \alpha > 0$$

which means that first inequality of (C) is not satisfied. Similarly, the assumption ‘ x is not right invertible in \mathcal{A}^{**} ’ implies that the second inequality of (C) is not satisfied (or just repeat the preceding argument with x^*). For the converse, now suppose x is invertible. Then x^*x is strictly positive element. More precisely, $x^*x \geq \lambda 1$, where λ equals the minimum value taken by x^*x (seen as a positive function). Therefore,

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 = \|x^*x + y^*y\| \geq \|\lambda 1 + y^*y\| = \lambda + \|y\|^2.$$

Once again, applying this computation to x^* , we can get the second inequality of (C). Note that both inequalities in (C) are valid for any $y \in \mathcal{A}^{**}$. Now suppose that α satisfies (C). Since $\lambda = 1/\|x^{-1}x^{*-1}\| = \|x^{-1}\|^{-2}$ (because the minimum value of a positive function is actually the inverse of the sup-norm of its inverse function), we just need to prove that $\alpha \leq \|x^{-1}\|^{-2}$. Suppose that $\alpha > \|x^{-1}\|^{-2}$. Then pick μ such that $\alpha > \mu > \|x^{-1}\|^{-2}$, so that $y = e_\mu$ (see notation of Theorem 5.2.2 in [9]), the spectral projection of x^*x corresponding to μ , is non-zero. Hence $\mu + 1 \geq \|x^*x + y^*y\| \geq \alpha + 1$, this is a contradiction. ■

Remark 2.3. Note that ‘ x invertible implies (C)’ is true in any unital operator algebras.

Proposition 2.4. *Let \mathcal{A} be a unital operator algebra and let \mathcal{B} be a unital C^* -algebra. Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a cb-isomorphism such that $\|T\|_{cb}\|T^{-1}\|_{cb} \leq 1 + \epsilon$, with $\epsilon < \sqrt{2} - 1$. Then $T(1)$ is invertible and*

$$\|T(1)^{-1}\| \leq \frac{1}{\|T\|_{cb}} \sqrt{\frac{(1 + \epsilon)^2}{2 - (1 + \epsilon)^2}}.$$

Proof. Replacing T by $T/\|T\|_{cb}$, we can assume that $\|T\|_{cb} = 1$ and $\|T^{-1}\|_{cb} < 1 + \epsilon$. Let $y \in \mathcal{B}^{**}$ be of norm one,

$$\left\| \begin{bmatrix} T(1) \\ y \end{bmatrix} \right\|^2 \geq \frac{1}{\|T^{-1}\|_{cb}^2} \left\| \begin{bmatrix} 1 \\ T^{-1**}(y) \end{bmatrix} \right\|^2 \geq \frac{1}{\|T^{-1}\|_{cb}^2} (1 + \|y\|^2)$$

the last inequality coming from the previous Lemma (applied to the unit of \mathcal{A}) and Remark 2.3. Now define $\alpha = \frac{2}{\|T^{-1}\|_{cb}^2} - 1$, which is strictly positive and satisfies (C) for any $y \in \mathcal{B}^{**}$ of norm one. Finally,

applying Lemma 2.2 to $T(1)$, we obtain

$$\|T(1)^{-1}\|^2 \leq 1/\alpha \leq \frac{(1+\epsilon)^2}{2-(1+\epsilon)^2}.$$

■

Corollary 2.5. *For each $d \in \mathbb{N}^*$, for any $\eta > 0$, there exists $\epsilon \in (0, \sqrt{2} - 1)$ with the following property: for any unital C^* -algebras \mathcal{A} , \mathcal{B} , for any cb-isomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$, $\|T\|_{cb} = 1$ and $T^{-1}\|_{cb} \leq 1 + \epsilon$ implies $\|L^{\vee d}\|_{cb} < \eta$, where $L = T(1)^{-1}T$.*

Proof. By Proposition 2.4, if ϵ is small enough, then $T(1)$ is invertible and so we can consider the linear map $L : \mathcal{A} \rightarrow \mathcal{B}$ defined by $L(x) = T(1)^{-1}T(x)$. We also have the estimates $\|L^{-1}\|_{cb} \leq 1 + \rho(\epsilon)$ and $\|L\|_{cb} \leq 1 + \rho(\epsilon)$ with $\rho(\epsilon)$ tending to 0 when ϵ tends to 0. Theorem 2.1 applied to L , then gives the result. ■

Remark 2.6. It would be nice to prove such a result for non-selfadjoint operator algebras.

3. APPLICATIONS

3.1. A non-commutative Amir-Cambern Theorem.

Definition 3.1. Let \mathcal{X} be an operator space. A bilinear map $m : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ is called a *multiplication on \mathcal{X}* if it is associative and extends to the Haagerup tensor product $\mathcal{X} \otimes_h \mathcal{X}$.

We denote by $m_{\mathcal{A}}$ the original multiplication on an operator algebra \mathcal{A} .

In the following, $H_{cb}^k(\mathcal{A}, \mathcal{A})$ denotes the k^{th} completely bounded cohomology group of \mathcal{A} over itself (see [15]).

The next proposition is the operator space version of Theorem 3 in [14]. The quantity $\|m - m_{\mathcal{A}}\|_{cb}$ is the cb-norm of $m - m_{\mathcal{A}}$ as a linear map from $\mathcal{A} \otimes_h \mathcal{A}$ into \mathcal{A} .

Proposition 3.2. *Let \mathcal{A} be an operator algebra satisfying*

$$H_{cb}^2(\mathcal{A}, \mathcal{A}) = H_{cb}^3(\mathcal{A}, \mathcal{A}) = 0. \quad (\star)$$

Then there exist $\delta, C > 0$ such that for every multiplication m on \mathcal{A} satisfying $\|m - m_{\mathcal{A}}\|_{cb} \leq \delta$, there is a completely bounded linear isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$\|\Phi - \text{id}_{\mathcal{A}}\|_{cb} \leq C\|m - m_{\mathcal{A}}\|_{cb} \text{ and } \Phi(m(x, y)) = \Phi(x)\Phi(y).$$

Moreover, if \mathcal{A} is a von Neumann algebra, then (\star) is necessarily satisfied and one can choose $\delta = 2^{-1}10^{-10}$ and $C = 4$.

Proof. One just has to apply the implicit function theorem (Theorem 1 in [14]) to the right spaces of completely bounded multilinear maps as in the proof of Theorem 3 in [14] (see also the proof of Theorem 7.4.1 in [15] for more details). Alternatively follow the proof of Theorem 4.1 [8], replacing norms of multilinear maps by their cb-norms. If \mathcal{A} is a von Neumann algebra, every completely bounded cohomology group of \mathcal{A} over itself vanish (see e.g. [15]) and we can choose $K = L = 1$ in the proof of Theorem 4.1 [8]. But after examination of the proof of Theorem 4.1 [8], we need $\delta < (2M_9^3)^{-1}$ and $C = M_4$, which proves the last assertion. ■

Now we are ready to prove the main result of this note.

Theorem 3.3. *There exists $\varepsilon_0 > 0$ such that for any von Neumann algebras \mathcal{M}, \mathcal{N} , the inequality $d_{cb}(\mathcal{M}, \mathcal{N}) < 1 + \varepsilon_0$ implies that \mathcal{M} and \mathcal{N} are $*$ -isomorphic.*

Proof. Let $T : \mathcal{M} \rightarrow \mathcal{N}$ be a linear cb-isomorphism between two von Neumann algebras \mathcal{M}, \mathcal{N} such that $\|T\|_{cb}\|T^{-1}\|_{cb} < 1 + \epsilon$, for some $\epsilon > 0$. Let us prove that if ϵ is small enough then $\mathcal{M} = \mathcal{N} *$ -isomorphically. If ϵ is small enough, by Corollary 2.5, we can consider the cb-isomorphism $L = T(1)^{-1}T$ satisfying $\|L^{-1}\|_{cb} \leq 1 + \rho(\epsilon)$ and $\|L\|_{cb} \leq 1 + \rho(\epsilon)$ with $\rho(\epsilon)$ tending to 0 when ϵ tends to 0. Then define a multiplication m on \mathcal{M} by

$$m(x, y) = L^{-1}(L(x)L(y)).$$

Note that

$$\|m - m_{\mathcal{M}}\|_{cb} \leq \|L^{-1}\|_{cb}\|L^{\vee 2}\|_{cb}.$$

But from Corollary 2.5, $\|L^{\vee 2}\|_{cb} < 2^{-2}10^{-10}$, if $\rho(\epsilon)$ is small enough. Finally, we obtain $\|m - m_{\mathcal{M}}\|_{cb} \leq 2^{-1}10^{-10}$ for ϵ small enough. By Proposition 3.2, this gives a linear cb-isomorphism $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ such that $\Phi(m(x, y)) = \Phi(x)\Phi(y)$. Therefore, $L \circ \Phi^{-1} : \mathcal{M} \rightarrow \mathcal{N}$ is an algebra isomorphism. By Theorem 3 in [6], it follows that \mathcal{M} and \mathcal{N} are $*$ -isomorphic. ■

Actually, the previous theorem remains valid if \mathcal{N} is only assumed to be a C^* -algebra (because Theorem 3 in [6] is true for C^* -algebras). Furthermore, in noticing that the previous proof used Proposition 2.4 only on one algebra, by Remark 2.3, we can state the following corollary.

Corollary 3.4. *There exists $\varepsilon_0 > 0$ such that the following holds: for any von Neumann algebra \mathcal{M} , for every unital operator algebra \mathcal{A} , the inequality $d_{cb}(\mathcal{M}, \mathcal{A}) < 1 + \varepsilon_0$ implies that \mathcal{M} and \mathcal{A} are cb-isomorphic via an algebra isomorphism.*

3.2. Stability of length. Our second application concerns the stability of length of a C^* -algebra. For details on the notion of length, see [11] and [12]. The first step is to notice that the length is stable for close multiplications, and then applying Corollary 2.5 we can prove the length is stable under perturbation by cb-isomorphisms with small bounds. In particular, it shows that the set of all C^* -algebras of length at most l is open inside the metric space of all C^* -algebras equipped with the Banach-Mazur cb-distance.

The next lemma is folklore in perturbation theory, we just sketch the proof.

Lemma 3.5. *Let $S, T : \mathcal{X} \rightarrow \mathcal{Y}$ be two completely bounded linear maps between operator spaces such that $\tilde{T} : \mathcal{X}/\ker T \rightarrow \mathcal{Y}$ is a cb-isomorphism with $\|\tilde{T}^{-1}\|_{cb} \leq K$. If $\|T - S\|_{cb} < 1/K$, then $\tilde{S} : \mathcal{X}/\ker S \rightarrow \mathcal{Y}$ is also a cb-isomorphism and*

$$\|\tilde{S}^{-1}\|_{cb} \leq \frac{K}{1 - K\|T - S\|_{cb}}.$$

Proof. Let y be in the unit ball of $\mathbb{M}_n(\mathcal{Y})$. Then there exists x_0 in $\mathbb{M}_n(\mathcal{X})$, $\|x_0\| < K$ such that $T(x_0) = y$. Hence

$$\|y - S(x_0)\| < \alpha,$$

where $\alpha = K\|T - S\|_{cb} < 1$. Applying the same procedure to $\frac{1}{\alpha}(y - S(x_0))$, we obtain x_1 in $\mathbb{M}_n(\mathcal{X})$, $\|x_1\| < K$ such that

$$\|y - S(x_0 + \alpha x_1)\| < \alpha^2$$

proceeding by induction, we obtain the result. ■

From [2] Theorem 5.2.1, we know that an operator space \mathcal{A} endowed with a multiplication m (in the sense of Definition 3.1) is cb-isomorphic via an algebra homomorphism to an actual operator algebra. As the length is invariant under algebraic cb-isomorphisms, it makes sense to talk about the length of \mathcal{A} equipped with m . We denote by m^l the l -linear map defined (by associativity) on \mathcal{A}^l .

Proposition 3.6. *Let \mathcal{A} be a unital operator algebra of length at most l and length constant at most K . Let m be another multiplication on \mathcal{A} such that $\|m^l - m_{\mathcal{A}}^l\|_{cb} < 1/K$. Then \mathcal{A} equipped with m has also length at most l .*

Proof. Let us denote by $T_l : \max(\mathcal{A})^{\otimes_n l} \rightarrow \mathcal{A}$ (resp. $S_l : \max(\mathcal{A})^{\otimes_h l} \rightarrow \mathcal{A}$) the completely bounded linear map induced by the original multiplication $m_{\mathcal{A}}$ (resp. by the new multiplication m) on the l -fold Haagerup tensor product of $\max(\mathcal{A})$ (i.e. \mathcal{A} endowed with its maximal operator

space structure). The hypothesis that \mathcal{A} has length at most l and length constant at most K exactly means that $\tilde{T}_l : \max(\mathcal{A})^{\otimes_{h^l}} / \ker T_l \rightarrow \mathcal{A}$ is a cb-isomorphism with $\|\tilde{T}_l^{-1}\|_{cb} \leq K$ (see [11] Theorem 4.2). But $\|m^l - m_{\mathcal{A}}^l\|_{cb} < 1/K$ implies that $\|T_l - S_l\|_{cb} < 1/K$. By Lemma 3.5, $\tilde{S}_l : \max(\mathcal{A})^{\otimes_{h^l}} / \ker S_l \rightarrow \mathcal{A}$ is also a cb-isomorphism, so the result follows. ■

Theorem 3.7. *Let $K \geq 1$ and $l \in \mathbb{N} \setminus \{0\}$ fixed but arbitrary constants. There exists $\varepsilon_{K,l} > 0$ with the following property: if \mathcal{A} and \mathcal{B} are unital C^* -algebras with $d_{cb}(\mathcal{A}, \mathcal{B}) < 1 + \varepsilon_{K,l}$ and \mathcal{A} of length at most l and length constant at most K , then \mathcal{B} has length at most l .*

Proof. As in the proof of Theorem 3.3, let $T : \mathcal{A} \rightarrow \mathcal{B}$ with $\|T\|_{cb}\|T^{-1}\|_{cb} < 1 + \epsilon$. We consider $L = T(1)^{-1}T$ and the multiplication m on \mathcal{A} defined by

$$m(x, y) = L^{-1}(L(x)L(y)).$$

Hence

$$\|m^l - m_{\mathcal{A}}^l\|_{cb} \leq \|L^{-1}\|_{cb}\|L^{\vee l}\|_{cb}.$$

Since $\|L^{-1}\|_{cb} \leq 1 + \rho(\epsilon)$ and $\|L\|_{cb} \leq 1 + \rho(\epsilon)$ with $\rho(\epsilon)$ tending to 0 when ϵ tends to 0, by Corollary 2.5, $\|L^{\vee l}\|_{cb} < K/2$, if $\rho(\epsilon)$ is small enough. Therefore by Proposition 3.6, \mathcal{A} equipped with m has also length at most l . But L is a cb-isomorphic algebra isomorphism from \mathcal{A} equipped with m onto \mathcal{B} , so \mathcal{B} has length at most l as well. ■

Remark 3.8. As in Corollary 3.4, one can note that the previous theorem is valid if \mathcal{B} is not self-adjoint.

Remark 3.9. We have just proved that the length is stable under cb-isomorphisms with small bound. More generally, any property which is stable under close multiplications is stable under cb-isomorphisms with small bound.

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